

Vector models of gravitational Lorentz symmetry breaking

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Spontaneous Lorentz symmetry breaking can occur when the dynamics of a tensor field cause it to take on a non-zero expectation value *in vacuo*, thereby providing one or more “preferred directions” in spacetime. Couplings between such fields and spacetime curvature will then affect the dynamics of the metric, leading to interesting gravitational effects. Bailey & Kostelecký [1] developed a post-Newtonian formalism that, under certain conditions concerning the field’s couplings and stress-energy, allows for the analysis of gravitational effects in the presence of Lorentz symmetry breaking. We perform a systematic survey of vector models of spontaneous Lorentz symmetry breaking. We find that a two-parameter class of vector models, those with kinetic terms we call “pseudo-Maxwell,” can be successfully analyzed under the Bailey-Kostelecký formalism, and that one of these two “dimensions” in parameter space has not yet been explored as a possible mechanism of spontaneous Lorentz symmetry breaking.

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I. INTRODUCTION

It is widely believed that classical general relativity, as formulated by Einstein, is a particular limit of some underlying theory of quantum gravity. However, at energy scales that are now accessible, it is expected (from our knowledge of effective field theory) that any fundamentally non-classical effects would be suppressed by at least a factor of the ratio of our experimental energy scale to the Planck scale; even for today’s most powerful particle colliders, this ratio still gives a suppression factor of 10^{-16} . With no foreseeable way to bridge this sixteen-order-of-magnitude gap in energy, we are forced to aim for sensitivity rather than power when searching for quantum-gravitational effects.

One particularly interesting avenue for this search is the possibility of quantum-suppressed Lorentz violation. In such a scenario, the underlying theory would include a tensor field (or fields) which spontaneously takes on a non-zero expectation value. Such a field would, in essence, provide a “preferred” direction or directions in spacetime.¹ The background value of this field could then couple weakly to conventional matter fields [2]; thus, the effects of such a tensor field could in principle be seen via careful observation of the behaviour of conventional particles and fields.

A particularly interesting venue in which to search for possible violations of Lorentz invariance is the gravita-

tional sector. Interactions between a dynamical metric and a tensor field with a non-zero expectation value have been postulated as a possible method of modifying cosmology [3, 4, 5, 6, 7], as a mechanism for modifying Newtonian gravity to solve the dark-matter problem [8, 9], or simply in their own right as modifications of conventional gravity [10, 11, 12]. Such modifications of gravity will, in general, cause modifications to the weak-field limit of gravity. The linearized effects of a direct coupling between Lorentz-violating fields and the Riemann tensor were analyzed in some detail by Bailey and Kostelecký [1]. By making certain assumptions about the properties of the equations of motion, they were able to obtain an effective linearized gravitational equation of the form

$$\delta G_{ab} + \mathfrak{T}_{ab}^{cdef} \delta R_{cdef} = 8\pi G \delta T_{ab} \quad (1)$$

where δG_{ab} and δR_{abcd} are (respectively) the Einstein and Riemann tensors linearized about a flat background, δT_{ab} is the stress-energy of conventional matter, and \mathfrak{T}_{ab}^{cdef} is a “small” tensor (in a sense we will make explicit below) depending in a particular way on the background values of the Lorentz-violating tensors. Using this effective equation, they then performed a thorough post-Newtonian analysis of such theories, examining the effects of Lorentz-violating fields on phenomena including satellite orbits, interferometric gravimetry, torsion-balance experiments, and frame-dragging.²

While this formalism is highly valuable for the analysis of the interface between gravity and Lorentz violation, its range of applicability is not immediately clear.

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¹ Such a field is often said to be “Lorentz-violating”. This description plays somewhat fast and loose with usual notions from the rest of physics; the field does, after all, transform as a proper tensor field under local Lorentz transformations. A more accurate way to describe such a field would be to say that it “spontaneously breaks Lorentz symmetry”, but such phrasing is rather more awkward. In the interests of readability and consistency with other papers in the literature, we will use “Lorentz-violating” in this sense as well.

² It is important to note that although the Bailey-Kostelecký formalism can be applied to the analysis of post-Newtonian gravity, the theories to which this formalism can be applied are in general not the same as those to which Will’s familiar Parametrized Post-Newtonian (PPN) formalism [13, 14] can be applied. The connections and distinctions between these two formalisms are explored in Section III C of Bailey and Kostelecký’s original paper [1].

To obtain the effective gravitational equation (1), it was necessary for Bailey and Kostelecký to place certain conditions on the equations of motion, rather than on the action from which they were derived. As action principles tend to be conceptually simpler than the equations of motion derived from them, it would be quite helpful to know whether a given action which includes spontaneous Lorentz symmetry breaking is analyzable in the Bailey-Kostelecký formalism. Should this be the case, the physical predictions of their paper [1] would be directly applicable to any such model.

This question is the focus of the present work. We will restrict our attention to the simplest type of tensor field which can spontaneously break Lorentz symmetry, namely vector fields A^a . In Section II, we describe the properties of the theories we will be concerned with, and we review the conditions required for successful use of the Bailey-Kostelecký formalism. Section III is dedicated to the application of these conditions to the vector actions under consideration; we will see that the class of vector theories for which the Bailey-Kostelecký formalism can successfully be used is not large, but that there do exist previously unconsidered models which can be analyzed in this framework. Finally, we discuss these results in Section IV.

We use the sign conventions of Wald [15] throughout, and units in which $c = 1$.

II. EQUATIONS OF MOTION AND FORMALISM

A. Actions for Lorentz-breaking vector fields

Bailey and Kostelecký's analysis of gravitational Lorentz violation [1] begins by assuming an action of the form

$$S = \int d^4x \sqrt{-g} (\mathcal{L}_{EH} + \mathcal{L}_{LV} + \mathcal{L}'). \quad (2)$$

\mathcal{L}_{EH} here is the usual Einstein-Hilbert action,

$$\mathcal{L}_{EH} = R - 2\Lambda. \quad (3)$$

We will assume throughout that $\Lambda = 0$. The second term, \mathcal{L}_{LV} , contains the non-trivial couplings of the Lorentz-violating fields to the metric:

$$\mathcal{L}_{LV} = -uR + s^{ab}(R^T)_{ab} + t^{abcd}C_{abcd}. \quad (4)$$

Here, R is the Ricci scalar, $(R^T)_{ab} \equiv R_{ab} - \frac{1}{4}g_{ab}R$ is the trace-free Ricci tensor, and C_{abcd} is the Weyl tensor. The tensors u , s^{ab} , and t^{abcd} may be fundamental fields or (as will be the case in our analysis) composites of other fields present in the theory. The final term, \mathcal{L}' , contains the terms determining the dynamics of the fundamental Lorentz-violating fields, as well as the action for conventional matter.

In the case of a single vector field being responsible for Lorentz symmetry breaking, we can be more specific in the form of the Lagrangian. Denoting the Lorentz-breaking vector field by A^a , the most general Lorentz-violation coupling terms will be of the form

$$\mathcal{L}_{LV} = \xi(-f_u(A^2)R + f_s(A^2)A^aA^bR_{ab}), \quad (5)$$

where $A^2 = A^aA_a$, f_u and f_s are arbitrary functions of A^2 , and ξ is a coupling constant.³ (By the symmetries of the Weyl tensor, any term analogous to $t^{abcd}C_{abcd}$ and constructed out of A^a and the metric must vanish.) This term is best thought of as a “weak” coupling term between the vector field and the curvature; the “weakness” of this coupling will be of importance in the next subsection.

The dynamics for A^a , meanwhile, will be determined by \mathcal{L}' . We can write the Lagrangian for an arbitrary second-differential-order vector theory as

$$\mathcal{L}' = K^a{}_b{}^c{}_d\nabla_aA^b\nabla_cA^d - V(A^2) + 2\kappa\mathcal{L}_{mat}, \quad (6)$$

where \mathcal{L}_{mat} is the Lagrangian for “conventional” matter; $\kappa = 8\pi G$; $V(A^2)$ is the potential for the vector field, constructed to have a minimum at a non-zero value of A^a ; and $K^a{}_b{}^c{}_d$ is a tensor constructed out of A^a and the metric. This tensor can be taken to be symmetric under the simultaneous exchange of $a \leftrightarrow c$ and $b \leftrightarrow d$. The conventional matter action \mathcal{L}_{mat} can, in principle, contain direct couplings to A^a . (We will introduce an explicit parametrization for $K^a{}_b{}^c{}_d$ in the next subsection.)

We can easily obtain the Euler-Lagrange equations associated with this action by varying the action with respect to g_{ab} and A^a ; there result the equations

$$(\mathcal{E}_g)^{ab} \equiv -G^{ab} + \xi\mathcal{A}^{ab} + \xi\mathcal{B}^{ab} + (T_A)^{ab} + \kappa(T_{mat})^{ab} = 0 \quad (7)$$

and

$$\begin{aligned} (\mathcal{E}_A)_a &\equiv 2\xi(-f'_uA_aR + f'_sA^bA^cR_{bc}A_a + f_sA^bR_{ba}) \\ &\quad + \mathcal{M}^b{}_c{}^d{}_{ea}\nabla_bA^c\nabla_dA^e - 2\nabla_b(K^b{}_a{}^c{}_d\nabla_cA^d) \\ &\quad - 2V'A_a + \frac{\delta\mathcal{L}_{mat}}{\delta A^a} = 0, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathcal{A}^{ab} &\equiv f_uG^{ab} + f'_uA^aA^bR \\ &\quad + \frac{1}{2}f_sg^{ab}A^cA^dR_{cd} + f'_sA^aA^bA^cA^dR_{cd}, \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{B}^{ab} &\equiv (g^{ab}\square - \nabla^a\nabla^b)f_u - \frac{1}{2}g^{ab}\nabla_c\nabla_d(f_sA^cA^d) \\ &\quad - \frac{1}{2}\square(f_sA^aA^b) + \nabla_c\nabla^{(a}(f_sA^{b)}A^c), \end{aligned} \quad (10)$$

³ Note that f_s is associated with the Ricci tensor in our parametrization, while in Bailey & Kostelecký's original paper the tensor s^{ab} is associated with the trace-free Ricci tensor.

$$(T_A)^{ab} \equiv \mathcal{M}^c{}_d{}^e{}_f{}^{ab} \nabla_c A^d \nabla_e A^f + \nabla_e \left((K^c{}_d{}^{(a|e|} A^{b)}) - K^c{}_d{}^{e(a} A^{b)} - K^{(ab)c}{}_d A^e) \nabla_c A^d \right) - \frac{1}{2} g^{ab} V - A^a A^b V', \quad (11)$$

$$(T_{\text{mat}})^{ab} \equiv \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\text{mat}})}{\delta g_{ab}}, \quad (12)$$

$$\mathcal{M}^c{}_d{}^e{}_f{}^{ab} \equiv \frac{1}{2} g^{ab} K^c{}_d{}^e{}_f + \frac{\delta K^c{}_d{}^e{}_f}{\delta g_{ab}}, \quad (13)$$

and

$$\mathcal{M}^b{}_c{}^d{}_e{}_a \equiv \frac{\delta K^b{}_c{}^d{}_e}{\delta A^a}. \quad (14)$$

(The arguments of the functions f_u , f_s , and V will be regularly omitted for brevity hereafter.)

B. Bailey-Kostelecký Formalism

The basic tack taken by Bailey and Kostelecký in their original paper [1] was to start from an action of the form (2), with its associated equations of motion; to construct the linearized equations of motion about a particular type of background; and to then impose certain conditions on the background and the equations of motion such that the linearized equations could be reduced to a particularly simple form:

$$\delta G_{ab} = \kappa(\delta T_{\text{mat}})_{ab} + \bar{u}\delta G_{ab} + \eta_{ab}\bar{s}^{cd}\delta R_{cd} - 2\bar{s}^c{}_{(a}\delta R_{b)c} + \frac{1}{2}\bar{s}_{ab}\delta R + \bar{s}^{cd}\delta R_{acdb}, \quad (15)$$

where \bar{u} and \bar{s}^{ab} are the background values of the fields u and s^{ab} . We now review and discuss these conditions as they pertain to the vector theories we are considering.

1. *The background values of the Lorentz-violating fields are constant with respect to a background flat spacetime.* In other words, if ϵ is our linearization parameter, we are looking for a family of solutions such that

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} \quad A^a = \bar{A}^a + \epsilon \tilde{A}^a \quad (16)$$

with $\bar{A}^a \neq 0$, and, in addition, that

$$\nabla_a A^b \sim \mathcal{O}(\epsilon). \quad (17)$$

We will see below that these requirements constrain the background values of V , as well as greatly simplifying the equations of motion (7) and (8).

2. *The dominant Lorentz-violating effects are linear in the vacuum values \bar{u} , \bar{s}^{ab} , and \bar{t}^{abcd} .* This can be enforced in our case by working only to linear

order in the coupling constant ξ , discarding terms of $\mathcal{O}(\xi^2)$ or higher. Turning this condition around, we will also require that in the limit of vanishing ξ , the metric will obey the Einstein equations; this ensures that our “Lorentz-violating” perturbed metric will only differ slightly from the usual perturbed metric derived from the conventional Einstein equations.

3. *The fluctuations \tilde{u} , \tilde{s}^{ab} , and \tilde{t}^{abcd} of the Lorentz-violating fields do not couple to the “conventional matter” sources.* This can be ensured by demanding that

$$\frac{\delta \mathcal{L}_{\text{mat}}}{\delta A^a} = 0, \quad (18)$$

thereby eliminating the last term from equation (8) above. In essence, this requirement ensures that it is only the metric that is directly affected by the dynamical Lorentz breaking. “Conventional” test particles will still move on geodesics with respect to the now-distorted metric, and these distorted paths can in principle allow us to indirectly observe the effects of Lorentz violation on gravity. In the remainder of this paper, we will be studying “vacuum solutions”, with all conventional matter sources set to zero.

4. *The independently conserved piece of the Lorentz-violating stress-energy $(T_A)^{ab}$ vanishes.* More specifically, if we take the divergence of the Einstein equation (7), we find that the divergence of $(T_A)^{ab}$ must equal the divergence of $\xi(A^{ab} + \mathcal{B}^{ab})$. This relation then allows us to “reverse-engineer” the form of $(T_A)^{ab}$, up to a piece Σ^{ab} whose divergence vanishes. This condition is then the statement that Σ^{ab} itself vanishes.⁴
5. *When the Einstein equation (7) is linearized, any second derivatives of \bar{A}^a can be eliminated from \mathcal{B}^{ab} and $(T_A)^{ab}$ in favour of second derivatives of the metric.* In practise, this elimination can only occur via the linearized vector equation of motion. This condition will be our primary focus in Section III.

As a consequence of the first condition above, the background (zero-order) equations of motion reduce simply to

$$\frac{1}{2} \eta^{ab} V(\bar{A}^2) + \bar{A}^a \bar{A}^b V'(\bar{A}^2) = 0 \quad (19)$$

and

$$V'(\bar{A}^2) \bar{A}_a = 0, \quad (20)$$

⁴ Note that this is not strictly speaking necessary for the analysis performed by Bailey and Kostelecký to still be valid, as noted in the original paper; in fact, it does not hold for the bumblebee model [1].

which together imply (as would be expected) that $V(\bar{A}^2) = V'(\bar{A}^2) = 0$. The linearized Einstein equation

of motion then becomes

$$\begin{aligned}\delta(\mathcal{E}_g)^{ab} &= -\delta G^{ab} + \xi \left(f_u \delta G^{ab} - f'_u \bar{A}^a \bar{A}^b \delta R + \frac{1}{2} f_s \eta^{ab} \bar{A}^c \bar{A}^d \delta R_{cd} + f'_s \bar{A}^a \bar{A}^b \bar{A}^c \bar{A}^d \delta R_{cd} \right) \\ &\quad + (\xi \mathcal{Q}_R^{abc}{}_d{}^e + \mathcal{Q}_K^{abc}{}_d{}^e) \delta(\nabla_e \nabla_c A^d) - V'' \bar{A}^a \bar{A}^b (2\bar{A}_c \bar{A}^c + h_{cd} \bar{A}^c \bar{A}^d),\end{aligned}\quad (21)$$

where

$$\begin{aligned}\mathcal{Q}_R^{abc}{}_d{}^e &\equiv 2f'_u (\eta^{ab} \eta^{ce} - \eta^{e(a} \eta^{b)c}) \bar{A}_d + f_s \left(-\eta^{ab} \bar{A}^{(c} \delta^{e)}_d - g^{ce} \bar{A}^{(a} \delta^{b)}_d + \eta^{c(a} \bar{A}^{b)} \delta^{e)}_d + \eta^{c(a} \delta^{b)}_d \bar{A}^e \right) \\ &\quad + f'_s (-\eta^{ab} \bar{A}^c \bar{A}^e - \bar{A}^a \bar{A}^b \eta^{ce} + 2\eta^{c(a} \bar{A}^{b)} \bar{A}^e) \bar{A}_d\end{aligned}\quad (22)$$

and

$$\mathcal{Q}_K^{abc}{}_d{}^e \equiv \bar{A}^{(a} K^{b)c}{}_d - \bar{A}^{(a} K^{[e|b)c}{}_d - \bar{A}^e K^{(ab)c}{}_d. \quad (23)$$

The linearized vector equation of motion, meanwhile, becomes

$$\begin{aligned}\frac{1}{2} \delta(\mathcal{E}_A)_a &= \xi (-f'_u \bar{A}_a \delta R + f'_s \bar{A}^b \bar{A}^c \delta R_{bc} \bar{A}_a + f_s \bar{A}^b \delta R_{ba} \\ &\quad - K^b{}_a{}^c{}_d \delta(\nabla_b \nabla_c A^d) - V'' \bar{A}_a (2\bar{A}_b \bar{A}^b + h_{bc} \bar{A}^b \bar{A}^c)).\end{aligned}\quad (24)$$

In equations (21)–(24), the arguments of the functions f_u , f_s , and V , as well as the tensor $K^a{}_b{}^c{}_d$, are understood to be evaluated at their background values $A^a \rightarrow \bar{A}^a$ and $g_{ab} \rightarrow \eta_{ab}$; indices are raised and lowered by the flat-space metric η_{ab} . The quantity $\delta(\nabla_a \nabla_b A^c)$ is given in terms of flat-space derivatives and the metric perturbation h_{ab} by

$$\delta(\nabla_a \nabla_b A^c) = \partial_a \left(\partial_b \tilde{A}^c + \left(\partial_{(b} h_{d)}{}^c - \frac{1}{2} \partial^c h_{bd} \right) \bar{A}^d \right). \quad (25)$$

Note that by Condition 1 above, this is an $\mathcal{O}(\epsilon)$ quantity.⁵ The quantities δR_{ab} , δG_{ab} , and $\delta R = \eta^{ab} \delta R_{ab}$, finally, are the linearized Ricci tensor, Einstein tensor, and Ricci scalar associated with the metric perturbation h_{ab} .

It will be to our advantage to introduce a concrete parametrization for the tensor $K^a{}_b{}^c{}_d$. Any tensor with the proper index structure constructed out of A^a and the

metric will be of the form

$$\begin{aligned}K^a{}_b{}^c{}_d &= \mathfrak{C}_1(A^2) g^{ac} g_{bd} + \mathfrak{C}_2(A^2) \delta^a{}_b \delta^c{}_d + \mathfrak{C}_3(A^2) \delta^a{}_d \delta^c{}_b \\ &\quad + \mathfrak{C}_4(A^2) A^a A^c g_{bd} + \frac{1}{2} \mathfrak{C}_5(A^2) (A^a A_d \delta^c{}_b + A^c A^b \delta^a{}_d) \\ &\quad + \mathfrak{C}_6(A^2) A_b A_d g^{ac} + \frac{1}{2} \mathfrak{C}_7(A^2) (A^a A_b \delta^c{}_d + A^c A_d \delta^a{}_b) \\ &\quad + \mathfrak{C}_8(A^2) A^a A^c A_b A_d,\end{aligned}\quad (26)$$

(This particular parametrization is due to Zlošnik *et al.* [9].) However, due to the geometric identity

$$\begin{aligned}\nabla^a &[f(A^2)(A_a \nabla_b A^b - A^b \nabla_a A_b)] \\ &= f(A^2) ((\nabla_a A^a)^2 - \nabla_b A^a \nabla_a A^b - R_{ab} A^a A^b) \\ &\quad + 2f'(A^2) (A^a A^c \nabla_a A_c \nabla_b A^b - A^a A^c \nabla_a A^b \nabla_b A_c),\end{aligned}\quad (27)$$

we can always eliminate one of \mathfrak{C}_2 , \mathfrak{C}_3 , \mathfrak{C}_5 , or \mathfrak{C}_7 via an integration by parts (thereby changing f_s as well.) Hereafter we will take \mathfrak{C}_2 to vanish. The arguments of $\mathfrak{C}_i(A^2)$ will also generally be omitted for brevity.

C. “Pseudo-Maxwell” kinetic terms

Finally, we note two important properties of the vector equation of motion (8) for certain choices of $K^a{}_b{}^c{}_d$. Consider a kinetic term for which $K^{(ab)c}{}_d = 0$. This places restrictions on the \mathfrak{C}_i functions:

$$\mathfrak{C}_1 + \mathfrak{C}_3 = 0 \quad (28a)$$

$$\mathfrak{C}_4 = -\frac{1}{2} \mathfrak{C}_5 = \mathfrak{C}_6 \quad (28b)$$

$$\mathfrak{C}_7 = \mathfrak{C}_8 = 0 \quad (28c)$$

Alternately, this condition implies a kinetic term that can be written in the form

$$\begin{aligned}K^a{}_b{}^c{}_d \nabla_a A^b \nabla_c A^d \\ = \pm (\mathcal{H}_1 g^{ac} + \mathcal{H}_2 A^a A^c) (\mathcal{H}_1 g^{bd} + \mathcal{H}_2 A^b A^d) F_{ab} F_{cd}\end{aligned}\quad (29)$$

⁵ It is also important to note that the flat-space derivative operator ∂_a and the covariant derivative operator ∇_a differ only at order ϵ . In particular, this means that the covariant derivative of an $\mathcal{O}(\epsilon)$ quantity (such as $\nabla_a A^b$) differs from its flat-space coordinate derivative by $\mathcal{O}(\epsilon^2)$, which for the purposes of this paper is negligible.

where $F_{ab} = 2\nabla_{[a}A_{b]}$, $\mathfrak{C}_1 = \pm\mathcal{H}_1^2$, and $\mathfrak{C}_4 = \pm\mathcal{H}_1\mathcal{H}_2$. (The signs here are determined by the overall sign of \mathfrak{C}_1 .) As this kinetic term is simply the familiar Maxwell field strength tensor contracted twice with a “generalized metric” $\mathcal{H}_1g^{ab} + \mathcal{H}_2A^aA^b$, we will call such kinetic terms (and theories containing them) “pseudo-Maxwell.”

Taking the divergence of the vector equation of motion (8) for a general $K^a{}_b{}^c{}_d$ and linearizing about our chosen background, we find that

$$\begin{aligned} & \xi(-f'_u\bar{A}^a\nabla_a\delta R + f'_s\bar{A}^b\bar{A}^c\bar{A}^a\nabla_a\delta R_{bc} + f_s\bar{A}^b\nabla^a\delta R_{ba}) \\ & - K^{ba}{}_d\delta(\nabla_a\nabla_b\nabla_c A^d) - 2V''(\bar{A}^2)\bar{A}^a\bar{A}_b\delta(\nabla_a A^b) = 0 \end{aligned} \quad (30)$$

For an arbitrary vector field A^a and an arbitrary metric, we know that

$$\begin{aligned} \nabla_a\nabla_b\nabla_c A^d &= \nabla_{(a}\nabla_{b)}(\nabla_c A^d) \\ &+ \frac{1}{2}(R_{abc}{}^e\nabla_e A^d - R_{abe}{}^d\nabla_c A^e). \end{aligned} \quad (31)$$

It can be then be seen that in the case $K^{(ba)c}{}_d = 0$, to linear order in ϵ the divergence of the vector equation of motion is simply

$$\begin{aligned} & \xi(-f'_u\bar{A}^a\nabla_a\delta R + f'_s\bar{A}^b\bar{A}^c\bar{A}^a\nabla_a\delta R_{bc} + f_s\bar{A}^b\nabla^a\delta R_{ba}) \\ & = 2V''(\bar{A}^2)\bar{A}^a\bar{A}_b\delta(\nabla_a A^b) \end{aligned} \quad (32)$$

(note that the quantity in brackets in equation (31) is $\mathcal{O}(\epsilon^2)$.) Using the linearized contracted Bianchi identity $\nabla^a\delta R_{ab} = \frac{1}{2}\nabla_b\delta R$, this last equation is equivalent to

$$\begin{aligned} \bar{A}^a\nabla_a\left(\xi\left(-f'_u + \frac{1}{2}f_s\right)\delta R + \xi f'_s\bar{A}^b\bar{A}^c\delta R_{bc}\right. \\ \left.- V''(\bar{A}^2)\delta(A^2)\right) = 0. \end{aligned} \quad (33)$$

where $\delta(A^2) = \delta(A^aA_a) = 2\tilde{A}^a\bar{A}_a + h_{ab}\bar{A}^a\bar{A}^b$.

This implies that in the case where $K^{(ab)c}{}_d = 0$, if the linearised quantity in brackets above vanishes on some hypersurface to which \bar{A}^a is non-tangent, this quantity will vanish throughout spacetime. (Recall that \bar{A}^a is a constant vector field in Minkowski space.) Thus, via an appropriate choice of boundary conditions, we can impose

$$\begin{aligned} \delta\mathcal{F} &\equiv \xi\left(-f'_u + \frac{1}{2}f_s\right)\delta R + \xi f'_s\bar{A}^b\bar{A}^c\delta R_{bc} \\ &- V''(\bar{A}^2)\delta(A^2) = 0 \end{aligned} \quad (34)$$

everywhere.⁶ This equation can be interpreted as telling us how much the vector field moves “up” its potential

(recall that the value of the potential V only depends on A^2), and so we will call the equation (34) the “massive-mode” condition. When combined with the linearized vector equations of motion (24), this yields

$$\frac{1}{2}\delta(\mathcal{E}_A)_a = \xi f_s\bar{A}^b\delta G_{ab} - K^b{}_a{}^c{}_d\delta(\nabla_b\nabla_c A^d) = 0. \quad (35)$$

This massive-mode condition can then be used to impose further conditions on A^a and its derivatives. It can be shown (see Appendix A) that by taking the appropriate combinations of the derivatives of the equation of motion, we arrive at the equation

$$\mathfrak{D}_a{}^b[\bar{A}^c\delta(\nabla_{[b}A_{c]})] = \xi f_s\bar{A}^b\bar{A}^c\partial_{[a}\delta G_{b]c} \quad (36)$$

where $\mathfrak{D}_a{}^b$ is the *flat-space* linear second-order differential operator

$$\begin{aligned} \mathfrak{D}_a{}^b &\equiv \mathfrak{C}_1\delta_a{}^b\square \\ &+ \mathfrak{C}_4(\delta_a{}^b\bar{A}^c\bar{A}^d\partial_c\partial_d + \bar{A}^2\partial_a\partial^b - \bar{A}_a\bar{A}^c\partial_c\partial^b). \end{aligned} \quad (37)$$

Thus, the operator $\mathfrak{D}_a{}^b$ applied to the one-form $v_a \equiv \bar{A}^b\delta(\nabla_{[a}A_{b]})$ yields a quantity of order ξ . The properties of $\mathfrak{D}_a{}^b$ (see Appendix A) allow us to conclude that under the imposition of appropriate boundary conditions, the quantity v_a will itself be of order ξ as long as

$$\mathfrak{C}_1(\mathfrak{C}_1 + \bar{A}^2\mathfrak{C}_4) > 0. \quad (38)$$

Since we also have

$$\bar{A}^a\delta(\nabla_b A_a) = \frac{1}{2}\delta(\nabla_b A^2) \sim \mathcal{O}(\xi) \quad (39)$$

from the massive-mode condition (34) above, we can conclude that under these assumptions, the quantity

$$\bar{A}^a\delta(\nabla_a A_b) = -2v_b + \frac{1}{2}\delta(\nabla_b A^2) \sim \mathcal{O}(\xi) \quad (40)$$

as well. This condition, along with the massive-mode condition (34), will become important in our analysis of the effective gravitational equations below.

III. CONDITIONS ON VECTOR DYNAMICS

A. The Einstein limit

1. General case

Recall the second of Bailey and Kostelecký’s conditions above: namely, that any Lorentz-violating corrections to the linearized Einstein equation are linear in the parameter ξ . This implies that in the limit $\xi \rightarrow 0$, the equations of motion (21) and (24) must together imply that the conventional linearized Einstein equation is satisfied, i.e., that $\delta G^{ab} = 0$. In this limit, the equations of motion become

$$-\delta G^{ab} + \mathcal{Q}_K{}^{abc}{}_d{}^e\delta(\nabla_e\nabla_c A^d) - V''\bar{A}^a\bar{A}^b\delta(A^2) = 0, \quad (41)$$

⁶ We have abused notation somewhat here, inasmuch as the quantity $\delta\mathcal{F}$ defined by (34) is not obtained as the linearized variation of some quantity \mathcal{F} . Nevertheless, we will continue to use $\delta\mathcal{F}$ throughout as a reminder that equations involving it are not exact, but only hold to linear order.

with \mathcal{Q}_K defined as in (23), and

$$-K^b{}_a{}^c{}_d \delta(\nabla_b \nabla_c A^d) - V'' \bar{A}_a \delta(A^2) = 0. \quad (42)$$

We will further allow the functions $\mathfrak{C}_i(A^2)$ to be dependent on ξ , defining functions $\mathcal{C}_i(A^2)$ and $\mathcal{D}_i(A^2)$ such that

$$\mathfrak{C}_i = \mathcal{C}_i + \xi \mathcal{D}_i + \mathcal{O}(\xi^2). \quad (43)$$

For the two equations (41) and (42) to imply the validity of the conventional linearized Einstein equation, we must be able to eliminate the terms containing second derivatives of the vector field from (41) using the vector equation of motion (42). Since this must occur for an arbitrary perturbation of the vector field, with arbitrary derivatives, we conclude that this will only occur if for some tensor \mathcal{T}^{abf} ,

$$\mathcal{Q}_K{}^{abc}{}_d{}^e = \mathcal{T}^{abf} K^e{}_f{}^c{}_d \quad (44)$$

in the limit $\xi \rightarrow 0$. If this relation holds, then we can combine the linearized Einstein equation and the linearized vector equation of motion to obtain

$$\delta G^{ab} = -V''(\bar{A}^2)(\bar{A}^a \bar{A}^b + \mathcal{T}^{abc} \bar{A}_c) \delta(A^2). \quad (45)$$

This further implies that if the conventional Einstein equation is to hold in the limit $\xi \rightarrow 0$, we must either have $\bar{A}^a \bar{A}^b + \mathcal{T}^{abc} \bar{A}_c = 0$ or $\delta(A^2) = 0$ in this limit.

What form must this tensor \mathcal{T}^{abc} have? For later convenience, we will split it up into pieces of $\mathcal{O}(\xi^0)$ and $\mathcal{O}(\xi^1)$:

$$\mathcal{T}^{abc} = \mathcal{T}_0^{abc} + \xi \hat{\mathcal{T}}^{abc}. \quad (46)$$

Moreover, since we are only concerned with the linearized equations, we can take \mathcal{T}^{abc} to be composed solely of background quantities. Since the only two geometric objects “in play” in the background are the vector field \bar{A}^a and the flat metric η^{ab} , and given the symmetry $\mathcal{T}^{abc} = \mathcal{T}^{bac}$ inherent in the definition of \mathcal{T}^{abc} , we conclude that \mathcal{T}^{abc} must be of the form

$$\mathcal{T}_0^{abc} = U_1 \eta^{ab} \bar{A}^c + U_2 \bar{A}^{(a} \eta^{b)c} + U_3 \bar{A}^a \bar{A}^b \bar{A}^c \quad (47)$$

and

$$\hat{\mathcal{T}}^{abc} = V_1 \eta^{ab} \bar{A}^c + V_2 \bar{A}^{(a} \eta^{b)c} + V_3 \bar{A}^a \bar{A}^b \bar{A}^c, \quad (48)$$

where the coefficients U_i and V_i can in principle be functions of \bar{A}^2 . Assuming that $\delta(A^2) \neq 0$, the constraint that $\bar{A}^a \bar{A}^b + \mathcal{T}_0^{abc} \bar{A}_c$ vanish yields:

$$U_1 = 0 \text{ and } U_2 + U_3 \bar{A}^2 + 1 = 0. \quad (49)$$

The question now becomes what form $K^a{}_b{}^c{}_d$ can have and still satisfy the condition (44). As with our other quantities, we will split $K^a{}_b{}^c{}_d$ into $\mathcal{O}(\xi^0)$ and $\mathcal{O}(\xi^1)$ parts:

$$K^a{}_b{}^c{}_d = (K_0)^a{}_b{}^c{}_d + \xi \hat{K}^a{}_b{}^c{}_d + \mathcal{O}(\xi^2). \quad (50)$$

Note that due to the decomposition (43), $(K_0)^a{}_b{}^c{}_d$ or $\hat{K}^a{}_b{}^c{}_d$ can be obtained by taking the original definition (26) of $K^a{}_b{}^c{}_d$ and replacing \mathfrak{C}_i by \mathcal{C}_i or \mathcal{D}_i , respectively. Similarly, we will define

$$\mathcal{Q}_K{}^{abc}{}_d{}^e = (\mathcal{Q}_{K0})^{abc}{}_d{}^e + \xi \hat{\mathcal{Q}}_K{}^{abc}{}_d{}^e + \mathcal{O}(\xi^2). \quad (51)$$

In the limit $\xi = 0$, we thus have the condition

$$(\mathcal{Q}_{K0})^{abc}{}_d{}^e = \mathcal{T}_0^{abf} (K_0)^e{}_f{}^c{}_d \quad (52)$$

Both sides of this equation consist of various five-index tensors constructed from \bar{A}^a and the metric, with various coefficients given in terms of U_2 and the \mathcal{C}_i functions. (Their exact forms are given in Appendix B, Equations (B1) and (B2).) Matching these coefficients, we obtain a set of eleven equations which the \mathcal{C}_i functions and U_2 must satisfy. (We of course want a non-trivial solution for the \mathcal{C}_i coefficients.) Examination of the resulting equations shows that we must have $U_2 = -2$ and $U_3 = \bar{A}^{-2}$, and that the functions \mathcal{C}_i must satisfy

$$C_1 = -C_3 = -\bar{A}^2 C_4 = \frac{1}{2} \bar{A}^2 C_5 \text{ and } C_7 = 0 \quad (53)$$

with C_6 and C_8 arbitrary. This implies a vector kinetic term that can be rewritten in the form

$$\begin{aligned} & K^a{}_b{}^c{}_d \nabla_a A^b \nabla_c A^d \\ &= \mathcal{G}_1(g^{ac} - A^{-2} A^a A^c)(g^{bd} - A^{-2} A^b A^d) F_{ab} F_{cd} \\ &+ (\mathcal{G}_2 g^{ab} + \mathcal{G}_3 A^a A^b) \nabla_a (A^2) \nabla_b (A^2) \end{aligned} \quad (54)$$

where $F_{ab} = 2 \nabla_{[a} A_{b]}$ and the coefficients \mathcal{G}_i are functions of A^2 , related to the \mathcal{C}_i functions by $C_1 = 2\mathcal{G}_1$, $C_6 = 4\mathcal{G}_2 - 2A^{-2}\mathcal{G}_1$, and $C_8 = 4\mathcal{G}_3$.

2. Pseudo-Maxwell dynamics

In the previous subsection, we assumed that a general form for $K^a{}_b{}^c{}_d$. However, as was noted at the end of Section II C, a “pseudo-Maxwell” vector kinetic term, satisfying $K^{(ab)}{}^c{}_d = 0$, will behave somewhat differently. The linearized solutions obtained from such an action will, with the imposition of appropriate boundary conditions, also meet additional self-consistency conditions due to properties of the linearized equations of motion. In particular, in the $\xi \rightarrow 0$ limit, the condition (34) becomes

$$V''(\bar{A}^2) \delta(A^2) = 0. \quad (55)$$

This allows us to ignore the constraints (49) on \mathcal{T}^{abc} , as they were imposed by the requirement that the right-hand side of Equation (45) vanish. We therefore only have the requirement that the second derivatives of A^a vanish, as expressed by (44), in order to obtain a valid Einstein limit. In this case, the full tensors are given by Equations (B3) and (B4) in Appendix B. Once again, we

perform the matching of coefficients between these two tensors, yielding a set of equations that must be satisfied by the \mathcal{C}_i and U_i functions. Assuming that $\mathcal{C}_1 \neq -\bar{A}^2\mathcal{C}_4$, these two tensors will be equal if and only if $U_2 = -2$ and $U_1 = U_3 = 0$.⁷ We have thus found two possible vector field kinetic terms, given by (29) and (54), for which the conventional Einstein limit is recovered in the limit of no direct coupling to curvature.

B. Adding Lorentz violation

In the above section, we obtained vector actions which satisfied Condition 2 above; namely, in the limit of no direct coupling to curvature, these actions yielded linearized equations of motion that implied the conventional linearized Einstein equation $\delta G^{ab} = 0$. We now wish to “turn on” direct coupling between the curvature and the vector field by setting $\xi \neq 0$ and place further constraints on the form of these actions.

Although Condition 2 does not yield any constraints on the form of the equations of motion at $\mathcal{O}(\xi)$, we can still constrain the vector action by imposing Condition 5: we must be able to eliminate the derivatives of A^a from the metric equation of motion (21) via use of the vector equation of motion (24). In particular, the terms in (21) which contain derivatives of the vector field can be written in the form

$$\begin{aligned} & (\mathcal{Q}_K{}^{abc}{}_d{}^e + \xi \mathcal{Q}_R{}^{abc}{}_d{}^e) \delta(\nabla_e \nabla_c A^d) \\ &= \left((\mathcal{Q}_{K0})^{abc}{}_d{}^e + \xi (\hat{\mathcal{Q}}_K{}^{abc}{}_d{}^e + \mathcal{Q}_R{}^{abc}{}_d{}^e) \right) \delta(\nabla_e \nabla_c A^d) \end{aligned} \quad (56)$$

Using the vector equation of motion (21) and the condition (44), we can rewrite this as

$$\begin{aligned} & (\mathcal{Q}_K{}^{abc}{}_d{}^e + \xi \mathcal{Q}_R{}^{abc}{}_d{}^e) \delta(\nabla_e \nabla_c A^d) \\ &\simeq \xi \left(-\mathcal{T}_0{}^{abf} \hat{K}{}^e{}_f{}_d + \hat{\mathcal{Q}}_K{}^{abc}{}_d{}^e + \mathcal{Q}_R{}^{abc}{}_d{}^e \right) \delta(\nabla_e \nabla_c A^d) \end{aligned} \quad (57)$$

where the “ \simeq ” symbol here means “up to terms not involving derivatives of A^a .” We can further simplify this expression by noting that in an arbitrary spacetime,

$$\nabla_a \nabla_b A^c = \nabla_{(a} \nabla_{b)} A^c - \frac{1}{2} R_{abd}{}^c A^d \quad (58)$$

or, in our case,

$$\delta(\nabla_a \nabla_b A^c) = \delta(\nabla_{(a} \nabla_{b)} A^c) - \frac{1}{2} \delta R_{abd}{}^c \bar{A}^d \quad (59)$$

up to linear order in ϵ . Thus, at $\mathcal{O}(\xi)$ we only need to eliminate the symmetrized second derivatives from the

metric equation of motion (21); the antisymmetrized second derivatives will merely result in contractions of \bar{A}^a with the linearized Riemann tensor, which are expected if the effective linearized gravitational equation is to be of the form (1). This will occur if $\hat{\mathcal{T}}^{abc}$ (the $\mathcal{O}(\xi)$ contribution to \mathcal{T}^{abc} defined in (46)) satisfies the equation

$$\mathcal{Q}_R{}^{ab(c}{}_d{}^e) + \hat{\mathcal{Q}}_K{}^{ab(c}{}_d{}^e) = \hat{\mathcal{T}}^{abf} K^{(e}{}_f{}^c)_d + \mathcal{T}_0{}^{abf} \hat{K}^{(e}{}_f{}^c)_d. \quad (60)$$

This equation is essentially the $\mathcal{O}(\xi)$ analog of Equation (44).

We can now proceed with the analysis of this equation as we did in the $\xi = 0$ limit: we write out the left-hand and right-hand sides in terms of various five-index tensors constructed from η^{ab} and \bar{A}^a , and match coefficients to determine the possible forms of the \mathcal{D}_i ’s and their corresponding \mathcal{T}^{abc} tensors. Expressions for the resulting tensors are given in Appendix B; the left-hand side of (60) is given by equation (B5), while the right-hand side is given by (B6).

1. General case

In the case where $K^{(ab)c}{}_d \neq 0$, we found in Section III A 1 that the kinetic terms for the vector must be given by (54), with $U_3 = \bar{A}^{-2}$. We now wish to match the coefficients in (B5) and (B6) to see what conditions can be placed on the \mathcal{D}_i coefficients and the functions f_s and f_u . Substituting in the appropriate relations for the \mathcal{C}_i ’s and U_3 , we find that if (B5) and (B6) are to agree, we are forced to set

$$f'_u(A^2) = 0 \quad (61)$$

and

$$f_s(A^2) = 0. \quad (62)$$

These conditions can most easily be seen from the coefficients of $\eta^{e(a} \eta^{b)c} \bar{A}_d$ and $\eta^{ab} \bar{A}^{(c} \delta^{e)}{}_d$, respectively. In other words, the vector model whose kinetic term is given by (54) cannot be modified with a Lorentz-violating curvature coupling of the form (5) and still satisfy the assumptions of the Bailey-Kostelecký formalism. (Note that setting $f_u(A^2)$ to a non-zero constant merely changes the effective value of G .) Thus, this theory cannot be successfully be analyzed under this formalism unless Lorentz-violating effects induced by the coupling term \mathcal{L}_{LV} vanish.

2. Pseudo-Maxwell dynamics

The obvious next step is to attempt the same coefficient matching for pseudo-Maxwell vector theories, as defined in (29). However, when we naïvely do so, we find that the same logic that forced us to abandon Lorentz violation in the vector model (54) again forces the Lorentz-violating functions f_u and f_s to vanish in the case of

⁷ Note that the case where $\mathcal{C}_1 = -\bar{A}^2\mathcal{C}_4$ is a special case of the kinetic term (54) derived in the previous section.

pseudo-Maxwell kinetic terms. This stands in opposition to the fact Bailey and Kostelecký successfully applied their formalism to the so-called “bumblebee model” [2] in their original paper [1]; the kinetic term for this model is the same as our pseudo-Maxwell kinetic term in the special case $\mathfrak{C}_1 = \text{constant}$ and $\mathfrak{C}_4 = 0$. What have we failed to take into account?

The missing pieces are the conditions on the linearized derivatives of A^a derived in Section II C. Namely, we found that under the imposition of certain boundary conditions, we have

$$\bar{A}^a \delta(\nabla_a A_b) \sim \bar{A}^a \delta(\nabla_b A_a) \sim \mathcal{O}(\xi) \quad (63)$$

everywhere in the spacetime. The role of these conditions is easiest to see by returning to Equation (57) and examining the $\mathcal{O}(\xi^1)$ derivative terms remaining in the equations of motion after eliminating the $\mathcal{O}(\xi^0)$ derivative terms. To wit, suppose there exist tensors \hat{T}^{abf} , c^{abc}_d , \tilde{c}^{abc}_d , and d^{abcd} such that we can write

$$\begin{aligned} & -\mathcal{T}_0^{abf} \hat{K}^e{}_f{}_d + \hat{\mathcal{Q}}_K^{abc}{}_d{}^e + \mathcal{Q}_R^{abc}{}_d{}^e \\ &= \hat{T}^{abf} (K_0)^e{}_f{}_d + c^{abc}{}_d \bar{A}^e + \tilde{c}^{abe}{}_d \bar{A}^c + d^{abce} \bar{A}_d \end{aligned} \quad (64)$$

The conditions (63) on the derivatives of A^a imply that to linear order in ϵ , $\bar{A}^c \delta(\nabla_e \nabla_c A^d)$ and $\bar{A}_d \delta(\nabla_e \nabla_c A^d)$ are of order ξ ; similarly, to this order in ϵ we will have

$$\begin{aligned} \bar{A}^e \delta(\nabla_e \nabla_c A^d) &= \bar{A}^e \delta(\nabla_c \nabla_e A^d) + \bar{A}^e \delta R_{cef}{}^d \bar{A}^f \\ &= \bar{A}^e \bar{A}^f \delta R_{cef}{}^d + \mathcal{O}(\xi). \end{aligned} \quad (65)$$

Thus, if Equation (64) holds, we will have

$$\begin{aligned} & \xi(-\mathcal{T}_0^{abf} \hat{K}^e{}_f{}_d + \hat{\mathcal{Q}}_K^{abc}{}_d{}^e + \mathcal{Q}_R^{abc}{}_d{}^e) \delta(\nabla_e \nabla_c A^d) \\ &= \xi(\hat{T}^{abf} (K_0)^e{}_f{}_d \delta(\nabla_e \nabla_c A^d) + c^{abc}{}_d \bar{A}^e \bar{A}^f R_{cef}{}^d) \\ & \quad + \mathcal{O}(\xi^2), \end{aligned} \quad (66)$$

since all the other terms on the right-hand side of (64) are of $\mathcal{O}(\xi)$ when contracted with $\delta(\nabla_e \nabla_c A^d)$.⁸ In essence, the derivative conditions (63) allow us to “ignore” certain of the equations arising from the coefficient-matching implicit in (60) at a given order in ξ .

To perform this decomposition, we first note that by taking the equation $\mathcal{T}_0^{abf} (K_0)^e{}_f{}_d = (\mathcal{Q}_{K0})^{abc}{}_d{}^e$ and replacing the \mathcal{C}_i functions with \mathcal{D}_i functions, we obtain

$$\mathcal{T}_0^{abf} \hat{K}^e{}_f{}_d = \hat{\mathcal{Q}}_K^{abc}{}_d{}^e. \quad (67)$$

(To put this another way, the relations (28) hold to all orders in ξ , and so $\mathcal{T}_0^{abf} K^e{}_f{}_d = \mathcal{Q}_K^{abc}{}_d{}^e$ to all orders.)

Thus, the first two terms on the left-hand side of (64) cancel, and we merely need to examine $\mathcal{Q}_R^{abc}{}_d{}^e$ to find out the required form of the tensors on the right-hand side. The form of $\mathcal{Q}_R^{abc}{}_d{}^e$ is given by (22); for a \hat{T}^{abf} given by (48), the quantity $\hat{T}^{abf} K^e{}_f{}_d$ is given by

$$\begin{aligned} \hat{T}^{abf} K^e{}_f{}_d &= V_2 \mathcal{C}_1 (\bar{A}^{(a} \delta^{b)}{}_d \eta^{ce} - \bar{A}^{(a} \eta^{b)c} \delta^{e)}{}_d) \\ &+ V_1 (\mathcal{C}_1 + \bar{A}^2 \mathcal{C}_4) (\eta^{ab} \eta^{ce} \bar{A}_d - \eta^{ab} \delta^{e)}{}_d \bar{A}^c) \\ &+ (V_2 \mathcal{C}_4 + V_3 (\mathcal{C}_1 + \bar{A}^2 \mathcal{C}_4)) \bar{A}^a \bar{A}^b (\eta^{ce} \bar{A}_d - \bar{A}^c \delta^{e)}{}_d) \\ &+ V_2 \mathcal{C}_4 (\bar{A}^{(a} \delta^{b)}{}_d \bar{A}^c \bar{A}^e - \bar{A}^{(a} \eta^{b)c} \bar{A}_d \bar{A}^e). \end{aligned} \quad (68)$$

Comparing these equations, we can then see that Equation (64) is satisfied if \hat{T}^{abc} has

$$V_2 \mathcal{C}_1 = -f_s, \quad (69)$$

with V_1 and V_3 arbitrary, and

$$c^{abc}{}_d = f_s \left(-\frac{1}{2} \eta^{ab} \delta^c{}_d + \eta^{c(a} \delta^{b)}{}_d \right). \quad (70)$$

Note that this latter quantity is independent of the form of \hat{T}^{abf} .

Finally, we confirm that the effective gravitational equations are of the proper form for these pseudo-Maxwell models. Applying the massive-mode condition (34) to the linearized Einstein equation (21), we obtain

$$\begin{aligned} \delta G^{ab} &= \xi \left(f_u \delta G^{ab} - \frac{1}{2} f_s (\bar{A}^a \bar{A}^b \delta R - \eta^{ab} \bar{A}^c \bar{A}^d \delta R_{cd}) \right) \\ &+ (\hat{\mathcal{Q}}_K^{abc}{}_d{}^e + \xi \mathcal{Q}_R^{abc}{}_d{}^e) \delta(\nabla_e \nabla_c A^d). \end{aligned} \quad (71)$$

Using the linearized vector equation of motion (35) contracted with $\mathcal{T}_0^{abf} = -2\bar{A}^{(a} \eta^{b)f}$, we can eliminate the $\mathcal{O}(\xi^0)$ derivative terms to obtain

$$\begin{aligned} \delta G^{ab} &= \xi \left(f_u \delta G^{ab} + \frac{1}{2} f_s \bar{A}^a \bar{A}^b \delta R - 2 f_s \bar{A}^{(a} \delta R^{b)}{}_c \bar{A}^c \right. \\ &\quad \left. + \frac{1}{2} f_s \eta^{ab} \bar{A}^c \bar{A}^d \delta R_{cd} + \mathcal{Q}_R^{abc}{}_d{}^e \delta(\nabla_e \nabla_c A^d) \right). \end{aligned} \quad (72)$$

Lastly, the remaining derivatives of A^a in the above equation can be eliminated using the derivative conditions, as noted above in equation (66); this yields

$$\begin{aligned} \delta G^{ab} &= \xi \left(f_u \delta G^{ab} + f_s \left(\frac{1}{2} \bar{A}^a \bar{A}^b \delta R - 2 \bar{A}^{(a} \delta R^{b)}{}_c \bar{A}^c \right. \right. \\ &\quad \left. \left. + \eta^{ab} \bar{A}^c \bar{A}^d \delta R_{cd} + \bar{A}^c \bar{A}^d \delta R^a{}_{cd}{}^b \right) \right). \end{aligned} \quad (73)$$

In our parametrization, the bumblebee model [2] is obtained by setting $f_s = 1$ and $f_u = 0$. Plugging in these values, this effective equation for δG^{ab} reduces to the form of the effective gravitational equation (15) found by Bailey and Kostelecký, with an “effective \bar{u} ” of $-\frac{3}{4} \bar{A}^2$ and with $\bar{s}^{ab} = \bar{A}^a \bar{A}^b - \frac{1}{4} \eta^{ab} \bar{A}^2$.

⁸ Note that the decomposition in (64) is ambiguous: it does not address what is to be done with terms of the form $C^{abc} \bar{A}_d \bar{A}^e$, for instance. However, it is easily seen from (66) that such terms will vanish when contracted with the Riemann tensor, so it does not matter whether we consider them to be part of c^{abc}_d or d^{abce} .

IV. DISCUSSION

We have systematically examined the dynamics of vector-tensor gravity theories with spontaneous Lorentz symmetry breaking. The primary constraints on the form of these theories were obtained by imposing two of Bailey & Kostelecký’s conditions: First, we required that the equations have the correct weak-field Einstein limit $\delta G_{ab} = 0$ when the Lorentz-violating terms (5) are “turned off” (Condition 2 of the list in Section II B); second, we required that the linearized stress-energy of the vector field vanish automatically when the linearized vector equations of motion held (Condition 5). The first of these requirements led us to the conclusion that the kinetic terms for our vector fields must be of the form (29) or (54). The vanishing of the linearized vector stress-energy was found to be a somewhat more subtle issue; we found that under the imposition of appropriate boundary conditions, the so-called pseudo-Maxwell vector models (those with kinetic terms of the form (29)) could lead to effective gravitational equations expressed solely in terms of the metric.

It is important to reiterate that the imposition of boundary conditions is necessary to obtain effective gravitational equations of the form used by Bailey and Kostelecký in their post-Newtonian analysis; as was noted at the beginning of Section III B 2, an arbitrary solution of the vector equations of motion will not have the proper relations between the derivatives of the vector field to cause the linearized vector stress-energy to vanish. In a certain sense, this confirms the aptness of the name “bumblebee model”. This name was originally inspired by the notion that according to received wisdom, bumblebees should not be able to fly; naïve calculations by engineers and entomologists in the 1930s seemed to show that the bumblebee’s wings were too small to allow it to fly, and only once more subtle aerodynamic effects were taken into account was the mystery explained. Similarly, a naïve comparison of the bumblebee vector equations of motion with its stress-energy causes us to conclude that we cannot introduce Lorentz-violating gravitational effects into the model; only once more subtle effects (namely, proper boundary conditions) are taken into account can Lorentz violation in the bumblebee model “fly.”

This said, the technique of imposing boundary conditions to obtain the desired effective gravitational equations is not entirely rigorous. In particular, we used the somewhat vague statement that “solutions depend continuously on initial data” to argue that the quantity $\bar{A}^a \delta(\nabla_a A_b)$ was of order ξ . While this is true, the notion of continuity associated with well-posedness of an initial value problem is defined in terms of the norms of the solutions on certain Sobolev spaces, and is not easy to gain a simple intuition about (see Chapter 10 of [15]). The notion of “continuous dependence on initial data” (and, by Duhamel’s principle, on sources) does allow us to say that we can always make $\bar{A}^a \delta(\nabla_a A_b)$ as small as we like

by tuning ξ to be “sufficiently small”; however, it is far from clear how small is “sufficient.” It would be instructive to obtain more careful estimates of how critically the magnitude of $\bar{A}^a \delta(\nabla_a A_b)$ depends on ξ ; however, such an analysis is well outside the scope of this paper.

In some sense, the fact that only pseudo-Maxwell kinetic terms are acceptable for Lorentz violation is not entirely surprising given the Bailey-Kostelecký formalism’s requirement of cancellations in the equations of motion. The quantity $\nabla_a A_b$ will, in general, depend both on derivatives of the vector field and derivatives of the metric (this latter dependence can be thought of as arising from the Christoffel symbols implicit in $\nabla_a A_b$.) A vector kinetic term containing an arbitrary contraction of $\nabla_a A_b$ with itself and other fields will then, in general, lead to a “cross term” between derivatives of the vector and derivatives of the metric in the kinetic terms of the theory [16]. However, the antisymmetrized derivative $\nabla_{[a} A_{b]}$ is independent of the metric, and so the kinetic terms for the metric and the vector will be decoupled when we contract $\nabla_{[a} A_{b]}$ with itself. It is therefore not surprising that this special property should have some bearing on the relation between the vector equations of motion and the gravitational equations of motion.

In the case of $\mathfrak{C}_4 = 0$ and \mathfrak{C}_1 constant, the pseudo-Maxwell theories we have been discussing become a simple Maxwell action for the vector field (albeit without gauge symmetry, which is broken by the presence of the potential.) However, the theories for which $\mathfrak{C}_4 \neq 0$ do not appear to have been previously considered in the literature, at least as far as concerns Lorentz-violating effects. In some sense, the presence of a $\mathfrak{C}_4 \neq 0$ term causes Lorentz violation for the Lorentz-violating field itself: at the linearized level, small perturbations of the vector field “see” the effective metric $\mathcal{H}_1 g^{ab} + \mathcal{H}_2 A^a A^b$ (as defined in (29)), rather than the spacetime metric g^{ab} . In particular, in the bumblebee model the Nambu-Goldstone modes of the Lorentz-violating vector field can be interpreted as a Maxwell field in a particular gauge [17]. If we naïvely extended this interpretation to a general pseudo-Maxwell theory, one would expect that the “speed of light” would be different from the “speed of gravity”, as the two fields would propagate on the null cones of two different metrics. Under such an interpretation the “photon” would almost certainly propagate anisotropically; it is also possible that such an interpretation would predict vacuum birefringence. Experimental bounds on such phenomena could then place bounds on the relative values of \mathcal{H}_1 and \mathcal{H}_2 . That said, this intuitional understanding may be complicated by the fact that the correspondence in the above-mentioned work [17] is in a non-standard gauge. It is also known that this correspondence does not carry over to theories with more general kinetic terms than the bumblebee model [18], though the class of models examined in this last work did not include the pseudo-Maxwell theories we have found. More work is needed to elucidate the correspondence (if any) between Maxwell theory and the Nambu-Goldstone modes of these new theories.

Finally, it is important to note that our results imply that the Bailey-Kostelecký formalism cannot successfully analyze theories with non-standard kinetic terms [4, 5, 8, 11]. This does not imply that post-Newtonian effects in such theories cannot be analyzed; in fact, Bailey and Kostelecký did precisely this in their original paper [1] for a Lagrangian identical to what Carroll *et al.* later called sigma-æther theory [4]. It is further possible that such a theory might in fact provide a viable model of Lorentz violation, consistent with current experimental constraints, even though it does not fit into the Bailey-Kostelecký formalism. In the absence of a more general

formalism for gravitational Lorentz violation, however, such theories will have to be analyzed on a case-by-case basis.

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APPENDIX A: DERIVATION AND HYPERBOLICITY OF THE OPERATOR \mathfrak{O}_a^b

Consider the following linearized combination of the vector equations of motion:

$$\frac{1}{2}\delta\left(A^b(\nabla_b(\mathcal{E}_A)_a - \nabla_a(\mathcal{E}_A)_b)\right) = -\bar{A}^f K^{dec}_{[a}\delta^{b]}_f \delta(\nabla_b\nabla_c\nabla_d A_e) - V''(\bar{A}^2)\bar{A}^b\bar{A}_{[a}\nabla_{b]}\delta(A^2) + \xi(-f'_u\bar{A}^b\bar{A}_{[a}\nabla_{b]}\delta R + f'_s\bar{A}^b\bar{A}^c\bar{A}^d\bar{A}_{[a}\nabla_{b]}\delta R_{cd} - f_s\bar{A}^b\bar{A}^c\nabla_{[a}\delta R_{b]c}) \quad (\text{A1})$$

Writing out the term $-\bar{A}^f K^{dec}_{[a}\delta^{b]}_f$ for a theory in which $K^{(ab)c}_d = 0$, we find

$$-\bar{A}^f K^{dec}_{[a}\delta^{b]}_f = \mathfrak{C}_1(3\bar{A}^{[b}\delta^d_a\eta^{e]c} + \delta^{[d}_a\bar{A}^{e]}\eta^{bc}) + \mathfrak{C}_4(\bar{A}^b\bar{A}^c\delta^{[d}_a\bar{A}^{e]} + \bar{A}^2\delta^b_a\eta^{c[d}\bar{A}^{e]} - \bar{A}_a\bar{A}^b\eta^{c[d}\bar{A}^{e]}) \quad (\text{A2})$$

Since $\delta(\nabla_{[b}\nabla_{c]}\nabla_d A_e) \sim \mathcal{O}(\epsilon^2)$ and $\nabla_{[a}\nabla_{b]}\nabla_d A_e = 0$, we can rewrite the first term on the right-hand side of (A1) (to linear order) as

$$-2\bar{A}^f K^{dec}_{[a}\delta^{b]}_f \delta(\nabla_b\nabla_c\nabla_d A_e) = -2\bar{A}^f K^{dec}_{[a}\delta^{b]}_f \delta(\nabla_c\nabla_b\nabla_d A_e) = 2\mathfrak{O}_a^d[\bar{A}^e\delta(\nabla_{[d}A_{e]})] \quad (\text{A3})$$

Further, applying the massive-mode condition $\delta\mathcal{F} = 0$, we can eliminate the term proportional to $V''(\bar{A}^2)$ from (A1), yielding

$$\mathfrak{O}_a^b[\bar{A}^c\delta(\nabla_{[b}A_{c]})] = \xi f_s\bar{A}^b\bar{A}^c\partial_{[a}\delta G_{b]c} \quad (\text{A4})$$

when the linearized vector equation of motion is satisfied.

Thus, the quantity $v_a \equiv \bar{A}^b\delta(\nabla_{[a}A_{b]})$ will satisfy a second-order differential equation (A4) in flat spacetime. Moreover, the source for this equation is “small”, i.e., of order ξ . We are thus led to the following question: under what conditions will the solution for v_a itself be of order ξ ? More precisely, let us pick some time coordinate t on Minkowski space. We know that if we set $\xi = 0$, $v_a = 0$ for all t is a valid solution of the Cauchy problem for (A4) with the boundary condition $v_a(t_0) = 0$ and $\partial v_a/\partial t|_{t_0} = 0$. We wish to know whether, as we “tune” ξ to zero, the solutions of v_a go “smoothly” to zero for these boundary conditions.

This is precisely the question of whether the operator \mathfrak{O}_a^b has a well-posed initial-value formulation.⁹ While

the general problem of whether an arbitrary operator possesses an initial-value formulation can be quite subtle, for operators in flat spacetime with constant coefficients (such as \mathfrak{O}_a^b) the situation is more clear-cut. Suppose \mathfrak{O}_a^b is a linear m^{th} -order differential operator which operates on N -tuples of functions in flat spacetime. (Thus, an equation of the form $\mathfrak{O}_a^b v_b = 0$ is a system of N linear m^{th} -order differential equations.) Associated with any such operator we can find an $N \times N$ polynomial-valued matrix $P_a^b(\lambda, \vec{\zeta})$ such that

$$P_a^b\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) = \mathfrak{O}_a^b, \quad (\text{A5})$$

i.e., if we take P_a^b and replace λ by $\partial/\partial t$ and $\vec{\zeta}$ by $\vec{\nabla}$, we obtain the operator \mathfrak{O}_a^b . We will further assume that the matrix P_a^b is constant with respect to space and time. It can then be shown [19, 20] that such an operator has a well-posed initial value formulation (with respect to an initial-data surface $t = \text{constant}$) if and only if there

⁹ Note that a “small” variation in the source terms in (A4) can be

mapped to a “small” variation in the boundary conditions via Duhamel’s principle.

exists a real number c such that the mN roots λ_i of the equation

$$\det \left[P(i\lambda, i\vec{\zeta}) \right] = 0 \quad (\text{A6})$$

satisfy $\Im(\lambda_i) > -c$ for all real vectors $\vec{\zeta}$. Such an operator is said to be “hyperbolic in the sense of Gårding.”

To apply this result to the case of the operator $\mathfrak{O}_a{}^b$, let us choose a Cartesian coordinate system on flat spacetime $\{t, x, y, z\}$ for which $\bar{A}^x = \bar{A}^y = 0$. Then the polynomial defined by (A6) becomes

$$(\mathfrak{C}_1 + \bar{A}^2 \mathfrak{C}_4)(\lambda^2 - \vec{\zeta}^2) \times \left(\mathfrak{C}_1(\lambda^2 - \vec{\zeta}^2) - \mathfrak{C}_4(\bar{A}^t \lambda + \bar{A}^z \zeta_3)^2 \right)^3 = 0 \quad (\text{A7})$$

This polynomial has roots when $\lambda_i^2 = \vec{\zeta}^2$ due to its second factor; these will obviously have $\Im(\lambda_i) = 0$ for all real $\vec{\zeta}$. The third factor, meanwhile, is a slightly more complicated quadratic polynomial in λ ; its roots can be shown to be real if its discriminant is positive:

$$\mathfrak{D} \equiv \mathfrak{C}_1 \left(\left(\mathfrak{C}_1 - \mathfrak{C}_4 (\bar{A}^t)^2 \right) \zeta_\perp^2 + (\mathfrak{C}_1 + \mathfrak{C}_4 \bar{A}^2) \zeta_z^2 \right) > 0, \quad (\text{A8})$$

where $\zeta_\perp^2 \equiv \zeta_x^2 + \zeta_y^2$. If the quantity \mathfrak{D} is negative for some value of $\vec{\zeta}$, the imaginary part of these roots will be $\pm\sqrt{\mathfrak{D}}$. Moreover, should this quantity \mathfrak{D} be negative for

some real vector $\vec{\zeta}$, the magnitude of the imaginary part of these roots can be made arbitrarily large: if $\Im(\lambda_i) = \pm\sqrt{\mathfrak{D}_0}$ for a given $\vec{\zeta} = \vec{\zeta}_0$, then $\Im(\lambda_i) = \pm M\sqrt{\mathfrak{D}_0}$ for $\vec{\zeta} = M\vec{\zeta}_0$. Thus, the operator $\mathfrak{O}_a{}^b$ defined in (37) will be hyperbolic in the sense of Gårding if and only if \mathfrak{D} is a positive definite quadratic form in $\vec{\zeta}$, i.e., if

$$\mathfrak{C}_1(\mathfrak{C}_1 - \mathfrak{C}_4 (\bar{A}^t)^2) > 0 \text{ and } \mathfrak{C}_1(\mathfrak{C}_1 + \mathfrak{C}_4 \bar{A}^2) > 0. \quad (\text{A9})$$

We can therefore conclude that in any frame in which these inequalities hold, we can then impose boundary conditions on some initial-time surface $t = t_0$ such that $\bar{A}^b \delta(\nabla_{[a} A_{b]}) \sim \mathcal{O}(\xi)$ throughout the spacetime. We can further ask that such a frame have $A^t \neq 0$; if this is the case, then the massive-mode condition (34) can also be imposed on the surface $t = t_0$, and it will follow (via the linearized equations of motion) that the massive-mode condition is satisfied everywhere. Such a frame will necessarily exist if

$$\mathfrak{C}_1(\mathfrak{C}_1 + \bar{A}^2 \mathfrak{C}_4) > 0. \quad (\text{A10})$$

(If $\bar{A}^2 < 0$, the frame in which $\bar{A}^z = 0$ satisfies our requirements; if $\bar{A}^2 \geq 0$, the required frame is one in which A^t is non-zero but sufficiently small that $\mathfrak{C}_1^2 > \mathfrak{C}_1 \mathfrak{C}_4 (\bar{A}^t)^2$.) For $\bar{A}^2 \neq 0$, this is equivalent to the condition that the “effective metric” appearing in (29) is of signature $(- +++)$ or $(+ --)$.

APPENDIX B: TENSOR COEFFICIENT-MATCHING

For a general vector theory, we will have

$$\begin{aligned} (\mathcal{Q}_{K0})^{abc}{}_d{}^e &= (\mathcal{C}_1 - \mathcal{C}_3) \bar{A}^{(a} \eta^{b)c} \delta^e{}_{d)} + (\mathcal{C}_3 - \mathcal{C}_1) \bar{A}^{(a} \delta^{b)}{}_{d)} \eta^{ce} - (\mathcal{C}_1 + \mathcal{C}_3) \delta^{(a}{}_d \eta^{b)c} \bar{A}^e \\ &\quad + \left(\mathcal{C}_4 - \frac{1}{2} \mathcal{C}_5 \right) \bar{A}^a \bar{A}^b \bar{A}^c \delta^e{}_{d)} - 2\mathcal{C}_4 \bar{A}^{(a} \delta^{b)}{}_{d)} \bar{A}^c \bar{A}^e + \left(\frac{1}{2} \mathcal{C}_5 - \mathcal{C}_6 \right) \bar{A}^a \bar{A}^b \eta^{ce} \bar{A}_d \\ &\quad - \mathcal{C}_5 \bar{A}^{(a} \eta^{b)c} \bar{A}_d \bar{A}^e - \frac{1}{2} \mathcal{C}_7 \bar{A}^a \bar{A}^b \delta^c{}_{d)} \bar{A}^e - \frac{1}{2} \mathcal{C}_7 \eta^{ab} \bar{A}^c \bar{A}_d \bar{A}^e - \mathcal{C}_8 \bar{A}^a \bar{A}^b \bar{A}^c \bar{A}_d \bar{A}^e. \end{aligned} \quad (\text{B1})$$

Assuming that $K^{(ab)c}{}_d \neq 0$, the tensor \mathcal{T}_0^{abc} must have $U_1 = 0$ and $U_2 + U_3 A^2 + 1 = 0$; multiplying these two tensors together, we find that

$$\begin{aligned} \mathcal{T}_0^{abf} (K_0)^e{}_f{}_d &= U_2 \mathcal{C}_3 \bar{A}^{(a} \eta^{b)c} \delta^e{}_{d)} + U_2 \mathcal{C}_1 \bar{A}^{(a} \delta^{b)}{}_{d)} \eta^{ce} - \left(\frac{1}{2} \mathcal{C}_5 + \bar{A}^{-2} (1 + U_2) \mathcal{C}_3 \right) \bar{A}^a \bar{A}^b \bar{A}^c \delta^e{}_{d)} \\ &\quad - \frac{1}{2} \mathcal{C}_7 \bar{A}^a \bar{A}^b \delta^c{}_{d)} \bar{A}^e + U_2 \mathcal{C}_4 \bar{A}^{(a} \delta^{b)}{}_{d)} \bar{A}^c \bar{A}^e - (\mathcal{C}_6 + \bar{A}^{-2} (1 + U_2) \mathcal{C}_1) \bar{A}^a \bar{A}^b \eta^{ce} \bar{A}_d + \frac{1}{2} U_2 \mathcal{C}_5 \bar{A}^{(a} \eta^{b)c} \bar{A}_d \bar{A}^e \\ &\quad + \frac{1}{2} U_2 \mathcal{C}_7 \bar{A}^{(a} \eta^{b)e} \bar{A}^c \bar{A}_d - \left(\mathcal{C}_8 + \bar{A}^{-2} (1 + U_2) \left(\mathcal{C}_4 + \frac{1}{2} (\mathcal{C}_5 + \mathcal{C}_7) \right) \right) \bar{A}^a \bar{A}^b \bar{A}^c \bar{A}_d \bar{A}^e. \end{aligned} \quad (\text{B2})$$

For a pseudo-Maxwell vector theory, we can obtain $(\mathcal{Q}_{K0})^{abc}{}_d{}^e$ simply by applying the conditions (28) to (B1); the result is

$$(\mathcal{Q}_{K0})^{abc}{}_d{}^e = 2\mathcal{C}_1 \bar{A}^{(a} \eta^{b)c} \delta^e{}_{d)} - 2\mathcal{C}_1 \bar{A}^{(a} \delta^{b)}{}_{d)} \eta^{ce} + 2\mathcal{C}_4 \bar{A}^a \bar{A}^b \bar{A}^c \delta^e{}_{d)} - 2\mathcal{C}_4 \bar{A}^{(a} \delta^{b)}{}_{d)} \bar{A}^c \bar{A}^e - 2\mathcal{C}_4 \bar{A}^a \bar{A}^b \eta^{ce} \bar{A}_d + 2\mathcal{C}_4 \bar{A}^{(a} \eta^{b)c} \bar{A}_d \bar{A}^e. \quad (\text{B3})$$

Due to the massive-mode condition, however, the above constraints on the functions U_i are relaxed; we thus must allow for arbitrary U_i functions, yielding

$$\begin{aligned} T_0^{abf}(K_0)^e{}_f{}_d = & U_2 \mathcal{C}_1 (\bar{A}^{(a} \delta^{b)}{}_d \eta^{ce} - \bar{A}^{(a} \eta^{b)c} \delta^e{}_d) + U_1 (\mathcal{C}_1 + \bar{A}^2 \mathcal{C}_4) \eta^{ab} (\eta^{ce} \bar{A}_d - A^c \delta^e{}_d) \\ & + (U_2 \mathcal{C}_4 + U_3 (\mathcal{C}_1 + \bar{A}^2 \mathcal{C}_4)) \bar{A}^a \bar{A}^b (\eta^{ce} \bar{A}_d - A^c \delta^e{}_d) + U_2 \mathcal{C}_4 (\bar{A}^{(a} \delta^{b)}{}_d \bar{A}^c \bar{A}^e - \bar{A}^{(a} \eta^{b)c} \bar{A}_d \bar{A}^e). \end{aligned} \quad (\text{B4})$$

At $\mathcal{O}(\xi)$, we can attempt an analogous coefficient matching for the tensors in Equation (60). The left-hand side of (60) is given by

$$\begin{aligned} \mathcal{Q}_R^{ab(c}{}_d{}^e} + \hat{\mathcal{Q}}_K^{ab(c}{}_d{}^e} = & (\mathcal{D}_1 - \mathcal{D}_3 + f_s) \bar{A}^{(a} \eta^{b)(c} \delta^e{}_d} + (\mathcal{D}_3 - \mathcal{D}_1 - f_s) \bar{A}^{(a} \delta^{b)}{}_d \eta^{ce} + (-\mathcal{D}_1 - \mathcal{D}_3 + f_s) \delta^{(a}{}_d \eta^{b)(c} \bar{A}^e) \\ & + 2f'_u \eta^{ab} \eta^{ce} \bar{A}_d - 2f'_u \eta^{e(a} \eta^{b)c} \bar{A}_d - f_s \eta^{ab} \bar{A}^{(c} \delta^e{}_d} + \left(\mathcal{D}_4 - \frac{1}{2} (\mathcal{D}_5 + \mathcal{D}_7) \right) \bar{A}^a \bar{A}^b \bar{A}^{(c} \delta^e{}_d} - 2\mathcal{D}_4 A^{(a} \delta^{b)}{}_d \bar{A}^c \bar{A}^e \\ & + \left(\frac{1}{2} \mathcal{D}_5 - \mathcal{D}_6 - f'_s \right) \bar{A}^a \bar{A}^b \eta^{ce} \bar{A}_d - (\mathcal{D}_5 - 2f'_s) \bar{A}^{(a} \eta^{b)(c} \bar{A}^e) \bar{A}_d - \left(\frac{1}{2} \mathcal{D}_7 + f'_s \right) \eta^{ab} \bar{A}^c \bar{A}^e \bar{A}_d - \mathcal{D}_8 \bar{A}^a \bar{A}^b \bar{A}^c \bar{A}^e \bar{A}_d, \end{aligned} \quad (\text{B5})$$

and the right-hand side is given by

$$\begin{aligned} (V_2 \mathcal{C}_3 - 2\mathcal{D}_3) \bar{A}^{(a} \eta^{b)(c} \delta^e{}_d} + (V_2 \mathcal{C}_1 - 2\mathcal{D}_1) \bar{A}^{(a} \delta^{b)}{}_d \eta^{ce} + V_1 (\mathcal{C}_1 + \bar{A}^2 \mathcal{C}_6) \eta^{ab} \eta^{ce} \bar{A}_d \\ + \left(\frac{1}{2} V_2 \mathcal{C}_5 - (\mathcal{D}_5 + \mathcal{D}_7) + V_3 \left(\mathcal{C}_3 + \frac{1}{2} \bar{A}^2 \mathcal{C}_5 \right) + U_3 \left(\mathcal{D}_3 + \frac{1}{2} \bar{A}^2 (\mathcal{D}_5 + \mathcal{D}_7) \right) \right) \bar{A}^a \bar{A}^b \bar{A}^{(c} \delta^e{}_d} \\ + V_1 \left(\mathcal{C}_3 + \frac{1}{2} \bar{A}^2 \mathcal{C}_5 \right) \eta^{ab} \bar{A}^{(c} \delta^e{}_d} + (V_2 \mathcal{C}_4 - 2\mathcal{D}_4) A^{(a} \delta^{b)}{}_d \bar{A}^c \bar{A}^e + \left(\frac{1}{2} V_2 \mathcal{C}_5 - \mathcal{D}_5 - \mathcal{D}_7 \right) \bar{A}^{(a} \eta^{b)(c} \bar{A}^e) \bar{A}_d \\ + (V_2 \mathcal{C}_6 - 2\mathcal{D}_6 + V_3 (\mathcal{C}_1 + \bar{A}^2 \mathcal{C}_6) + U_3 (\mathcal{D}_1 + \bar{A}^2 \mathcal{D}_6)) \bar{A}^a \bar{A}^b \eta^{ce} \bar{A}_d + V_1 \left(\mathcal{C}_4 + \frac{1}{2} \mathcal{C}_5 + \bar{A}^2 \mathcal{C}_8 \right) \eta^{ab} \bar{A}^c \bar{A}^e \bar{A}_d \\ + \left(V_2 \mathcal{C}_8 - 2\mathcal{D}_8 + V_3 \left(\mathcal{C}_4 + \frac{1}{2} \mathcal{C}_5 + \bar{A}^2 \mathcal{C}_8 \right) + U_3 \left(\mathcal{D}_4 + \frac{1}{2} (\mathcal{D}_5 + \mathcal{D}_7) + \bar{A}^2 \mathcal{D}_8 \right) \right) \bar{A}^a \bar{A}^b \bar{A}^c \bar{A}^e \bar{A}_d. \end{aligned} \quad (\text{B6})$$

We have used the fact that both candidate vector kinetic terms found in the previous section have $U_1 = 0$, $U_2 = -2$ and $\mathcal{C}_7 = 0$.

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